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# Periodic optimal control for competing parabolic Volterra–Lotka-type systems

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**Abstract**

In this paper, we study the optimal harvesting control problem governed by a time-periodic competing parabolic Volterra–Lotka system. We show the existence of an optimal control, and we also find some conditions which enable the characterization of the optimal control in terms of a large parabolic optimality system. We further construct monotone sequences closing in to all appropriate solutions of the periodic optimality system.

*Keywords:* Optimal control; Alternating monotone iterations

## 1. Introduction and statement of the problem

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$ boundary, $G = \Omega \times [0, T)$, $S = \partial\Omega \times [0, T)$ for some $T > 0$, and $b_i$, $c_i$ some positive constants, $i = 1, 2$. Throughout this paper we will always assume that $f(x, t)$, $g(x, t)$ and $a_i(x, t)$, $i = 1, 2$, are functions satisfying

$$f,\ g,\ a_i,\ i = 1, 2, \in L^\infty_+(\Omega \times (-\infty, \infty)) = \{h \in L^\infty(\Omega \times (-\infty, \infty)) \mid h \geqslant 0 \text{ in } G\},$$

and they are periodic functions of $t$ with period $T$ for $(x, t) \in \Omega \times (-\infty, \infty)$.

For any constant vector $\boldsymbol{\delta} = (\delta_1, \delta_2)$, $\delta_i > 0$, $i = 1, 2$, we let

$$\boldsymbol{B}_{\delta,T} = \{(f, g) \mid f,\ g \in L^\infty_+(\Omega \times (-\infty, \infty)),$$
$$f \text{ and } g \text{ are periodic functions of } t \text{ with period } T, \text{ and } f \leqslant \delta_1,\ g \leqslant \delta_2\}.$$

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For any $(f, g) \in \boldsymbol{B}_{\delta,T}$, we define $(u, v) = (u(f, g), v(f, g))$ as a solution of the problem

$$\begin{cases} u_t - \Delta u - u[(a_1 - f) - b_1 u - c_1 v] = 0, & \text{in } G, \\ v_t - \Delta v - v[(a_2 - g) - b_2 v - c_2 u] = 0, & \text{in } G, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on } S, \\ u(x, 0) = u(x, T) \text{ and } v(x, 0) = v(x, T), & \text{for } x \in \Omega. \end{cases} \tag{1.1}$$

We will show that such $(u(f, g), v(f, g))$ is uniquely defined when $\delta_i$, $a_i$, $b_i$, $c_i$, $i = 1, 2$, satisfy appropriate conditions (cf. (H1), (H2) below).

Next, let $K_i$, $M_i$, $i = 1, 2$, be positive constants; we define the pay-off function by

$$J(f, g) = \int_G [K_1 u(f, g) f + K_2 v(f, g) g - M_1 f^2 - M_2 g^2] \,\mathrm{d}x \,\mathrm{d}t, \tag{1.2}$$

which describes the economical return of harvesting the competing species $u$, $v$.

The problem is to find the periodic control $(f, g) \in \boldsymbol{B}_{\delta,T}$, such that

$$J(f, g) = \sup_{(f,g) \in \boldsymbol{B}_{\delta,T}} J(f, g). \tag{1.3}$$

In practical terms, we are searching for optimal harvesting of two competing biological species whose growth are governed by the diffusive Volterra–Lotka-type system (1.1). Here $a_i(x, t)$, $i = 1, 2$, describes spatially dependent intrinsic growth, $b_i$, $i = 1, 2$, designates crowding effect and the functions $f$ and $g$ denote distributions of control harvesting effort on the biological species. The optimal control criterion is to maximize the pay-off functional, where $K_1$ and $M_1$ are constants describing the market price of species $u$ and the cost of control $f$, and similarly $K_2$ and $M_2$ are constants related to $v$ and $g$.

In Section 2, we discuss the existence and uniqueness of a positive solution to (1.1). Then we prove the existence of optimal control for our problem. In Section 3, we find some conditions which enable us to characterize an optimal control in terms of solution of a parabolic optimality system. In Section 4 we construct monotone sequences closing in to all appropriate solutions of the optimality system. If the monotone increasing and decreasing sequences converge to a same function, then the optimal control is unique. Similar problems have been studied for the elliptic case in [11,13]. The one parabolic equation case is considered in [5]. The results here are different from [13], because the species interact differently and the system is now time-dependent. Other related results, regarding monotone iteration techniques in optimal control and game theory for partial differential equations, can be found in [9,10,12,15,16]. Many recent results concerning periodic solutions of competing systems can be found in, e.g., [1,2,6,8].

Here we shall use the stardard notation (see [5]) $W_p^{2,1}(G)$, and $L^p(G)$ for Sobolev space and $L^p$-space on $G = \Omega \times [0, T)$. $L_+^p(G) = \{f \in L^p(G) \mid f \geqslant 0$ a.e. in $G\}$. For convenience we will denote the norm in $L^p(G)$ by $\| \cdot \|_{p,G}$.

## 2. Existence of positive solution and optimal control

We first consider the existence of periodic positive solution to (1.1) for a fixed given $(f, g) \in \boldsymbol{B}_{\delta,T}$. This will be established in Theorem 2.2 with the hypothesis

$$\text{(H1)} \qquad 0 < \delta_i < \frac{1}{3}\left\{2\tilde{a}_i - \hat{a}_i - 2c_i\frac{\hat{a}_j}{b_j}\right\}, \quad i, j = 1, 2 \text{ and } i \neq j.$$

Here $\tilde{a}_i = \inf_G a_i(x, t)$ and $\hat{a}_i = \sup_G a_i(x, t)$ for $i = 1, 2$.

To obtain the existence Theorem 2.2, we will construct two sequences by means of iteration. Let $u_0$ be the solution of the problem

$$\begin{cases} u_t - \Delta u - (a_1 - f)u + b_1 u^2 = 0, & \text{in } G, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } S, \\ u(x, 0) = u(x, T), & \text{for } x \in \Omega, \end{cases} \tag{2.1}$$

and let $v_0$ be the solution of the problem

$$\begin{cases} u_t - \Delta u(a_2 - g - c_2 u_0)u + b_2 u^2 = 0, & \text{in } G, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } S, \\ u(x, 0) = u(x, T), & \text{for } x \in \Omega. \end{cases} \tag{2.2}$$

First, [5, Theorem 2.4] and (H1) imply that $u_0$ and $v_0$ exist in $W_p^{2,1}(G)$ for $p > 1$, and

$$\frac{\hat{a}_1}{b_1} \geqslant u_0 \geqslant \frac{\tilde{a}_1 - \delta_1}{b_1} > 0, \qquad \frac{\hat{a}_2}{b_2} \geqslant v_0 \geqslant \frac{\tilde{a}_2 - c_2\hat{a}_1/b_1 - \delta_2}{b_2} > 0.$$

(Note that in (2.1), (2.2), the derivatives are taken in the weak sense and the equations are satisfied a.e. in $G$. All solutions will be interpreted this way unless otherwise stated. For more details, see [5,7].)

For $i = 1, 2, \ldots,$ we define $u_i$ and $v_i$ as the solutions of the following problems (2.3) and (2.4), respectively:

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \Delta u_i - (a_i - f - c_1 v_{i-1})u_i + b_1 u_1^2 = 0, & \text{in } G, \\ \dfrac{\partial u_i}{\partial \nu} = 0, & \text{on } S, \\ u_i(x, 0) = u(x, T), & \text{for } x \in \Omega, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \dfrac{\partial v_i}{\partial t} - \Delta v_i - (a_2 - g - c_2 u_i) v_i + b_2 v_i^2 = 0, & \text{in } G, \\ \dfrac{\partial v_i}{\partial \nu} = 0, & \text{on } S, \\ v_i(x, 0) = v_i(x, T), & \text{for } x \in \Omega, \end{cases} \tag{2.4}$$

where $S = \partial\Omega \times [0, T)$ and $\nu$ is the outnormal vector on $S$.

Inductively, from (2.3) and (2.4), by [5, (2.28) and (2.29)], we obtain

$$0 < \frac{\tilde{a}_1 - c_1(\hat{a}_2/b_2) - \delta_1}{b_1} \leqslant u_k \leqslant \frac{\hat{a}_1}{b_1}, \qquad 0 < \frac{\tilde{a}_2 - c_2(\hat{a}_1/b_1) - \delta_2}{b_2} \leqslant v_k \leqslant \frac{\hat{a}_2}{b_2}. \tag{2.5}$$

Moreover, $u_k$ and $v_k$ are in $W_p^{2,1}(G)$, for $p > 1$, $k = 1, 2, \ldots$.

**Lemma 2.1.** *If the hypothesis* (H1) *holds, then the sequences* $\{u_k\}$ *and* $\{v_k\}$ *satisfy*

$$u_0 \geqslant u_1 \geqslant u_2 \geqslant \cdots \geqslant u_k \geqslant \cdots, \quad \text{in } G, \tag{2.6}$$

*and*

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant \cdots \leqslant v_k \leqslant \cdots, \quad \text{in } G. \tag{2.7}$$

**Proof.** First we should notice that the comparison lemma [5, Lemma 2.3] can be extended to include the case $c \in L_+^\infty(\Omega \times (-\infty, \infty))$. In fact, suppose $w_i$ is the solution of the problem

$$\begin{cases} w_t - \Delta w + cw = f_1, & \text{in } G, \\ \dfrac{\partial w}{\partial \nu} = 0, & \text{on } S, \\ w(x, 0) = w(x, T), & \text{for } x \in \Omega, \end{cases} \tag{2.8}$$

for $i = 1, 2$, where $c \in L_+^\infty(\Omega \times (-\infty, \infty))$. We need to prove that $f_1 \geqslant f_2$, $f_i \in L^p(G)$ for $i = 1, 2$, implies $w_1 \geqslant w_2$ in $G$.

Let $c_n \in C^\infty(\Omega \times (-\infty, \infty))$, $n = 1, 2, \ldots$, be periodic in $t$ with period $T$ such that

$$c_n \to c, \quad \text{in } L^p(G), \quad \text{as } n \to \infty.$$

Moreover, let $w_{in}$ be the solution of problem (2.8) with $c$ replaced by $c_n$ for $i = 1, 2$. Then [5, Lemma 2.3] implies that

$$w_{1n} \geqslant w_{2n}, \quad \text{in } G, \tag{2.9}$$

Now we only need to prove that $w_{in} \to w_i$ a.e. in $G$ as $n \to \infty$, $i = 1, 2$. But [5, Theorem 2.2] implies that $\{w_{in}\}$ is uniformly bounded in $W_p^{2,1}(G)$ for each $i = 1, 2$. Hence, using the same argument as in the proof of [5, Lemma 3.1], we can readily obtain $w_{in} \to w_i$ a.e. in $G$ as $n \to \infty$ for $i = 1, 2$. Therefore, from the inequality (2.9), we obtain $w_1 \geqslant w_2$.

We are now ready to prove (2.6) and (2.7). Let $w = u_0 - u_1$; then $w$ satisfies the inequality

$$w_t - \Delta w + [b_1(u_0 + u_1) - (a_1 - f) + c_1 v_0] w \geqslant 0. \tag{2.10}$$

The hypothesis (H1) and (2.5) imply that

$$b_1(u_0+u_1)-(a_1-f)+c_1v_0 \geqslant \delta > 0, \quad \text{in } G.$$

Then by the above extension of [5, Lemma 2.3], we conclude that $w \geqslant 0$, i.e.,

$$u_0 \geqslant u_1, \quad \text{in } G. \tag{2.11}$$

From this we deduce that $(v_1-v_0)$ satisfies the inequality

$$(v_1-v_0)_t - \Delta(v_1-v_0) + [b_2(v_1+v_0)-(a_2-g)+c_2u_0](v_1-v_0) \geqslant 0,$$

in $G$. The same argument as above implies that $v_1 \geqslant v_0$ in $G$.

By iterating and induction in $k$, we deduce by the same argument that (2.6) and (2.7) hold. □

By (2.5)–(2.7) and [5, Theorem 2.4], we obtain the estimates

$$\| u_k \|_{W_p^{2,1}(G)} \leqslant R_1, \qquad \| v_k \|_{W_p^{2,1}(G)} \leqslant R_2, \tag{2.12}$$

where $R_1$ and $R_2$ are constants independent of $k$.

By a similar argument as in [5, Theorem 2.4], taking the limit as $i \to \infty$ in (2.6) and (2.7) and using the a priori estimates (2.12), we finally conclude that there exists a solution $(u, v)$ of problem (2.1) in $W_p^{2,1}(G) \times W_p^{2,1}(G)$ and the estimates (2.12) for $u$ and $v$ hold. Hence we have proved the following theorem.

**Theorem 2.2.** *Suppose the hypothesis* (H1) *holds. Then the problem* (1.1) *has a solution in* $W_p^{2,1}(G) \times W_p^{2,1}(G)$ *for* $p > 1$ *with* $u$, $v$ *satisfying*

$$0 < \epsilon_1 \leqslant u \leqslant C_1, \qquad 0 < \epsilon_2 \leqslant v \leqslant C_2. \tag{2.13}$$

*Here* $\epsilon_i = [\tilde{a}_i - c_i(\hat{a}_j/b_j) - \delta_i]/b_i$ *and* $C_i = \hat{a}_i/b_i$, $i = 1, 2$ *and* $i \neq j$. *Moreover*, $(u, v)$ *satisfies*

$$\| u \|_{W_p^{2,1}(G)} \leqslant R_1, \qquad \| v \|_{W_p^{2,1}(G)} \leqslant R_2. \tag{2.14}$$

*Here* $R_i$ *is a constant determined by* $\| a_i \|_{\infty,G}$, $i = 1, 2$, *respectively*.

In order to obtain uniqueness of solution to problem (1.1), we introduce the following hypothesis:

(H2) $$c_i\frac{\hat{a}_2}{b_2} + c_j\frac{\hat{a}_1}{b_1} \leqslant 2\min\{\delta_1, \delta_2\}, \quad \text{for } i, j = 1, 2 \text{ and } i \neq j.$$

**Theorem 2.3.** *Let* $\delta_i$, $a_i$, $c_i$ *and* $b_i$, $i = 1, 2$, *satisfy the hypotheses* (H1) *and* (H2). *Then the problem* (1.1) *has a unique solution* $(u, v)$ *in* $W_p^{2,1}(G) \times W_p^{2,1}(G)$ *for* $p > 1$ *with* $u, v > 0$.

**Proof.** We first prove that if $(u, v)$ is a solution of problem (1.1) with $u, v > 0$, then $u$ and $v$ satisfy (2.13). In fact, we can use the same comparison lemma described in the proof of Lemma 2.1 to prove $u_0 \geqslant u$, and then $v_0 \leqslant v$. Similarly, we can show $u_k \geqslant u$ and $v_k \leqslant v$ for all $k = 0, 1, 2, \ldots$. Finally, we obtain the inequalities $u \leqslant \lim_{k\to\infty} u_k \leqslant C_1$ and $v \geqslant \lim_{k\to\infty} v_k \geqslant \epsilon_2$.

Interchanging the role of $u$ and $v$, we can show by means of symmetry that $u \geqslant \epsilon_1$ and $v \leqslant C_2$.

Suppose that there exist two solutions $(u_1, v_1)$ and $(u_2, v_2)$ of problem (1.1) with $u_i, v_i > 0$ for $i = 1, 2$; then $(u_1 - u_2, v_1 - v_2)$ satisfies

$$\begin{aligned}
&(u_1 - u_2)_t - \Delta(u_1 - u_2) + [b_1(u_1 + u_2) - (a_1 - f)](u_1 - u_2) + c_1 v_1(u_1 - u_2) \\
&\quad + c_1 u_2(v_1 - v_2) = 0, \quad \text{in } G, \\
&(v_1 - v_2)_t - \Delta(v_1 - v_2) + [b_2(v_1 + v_2) - (a_2 - g)](v_1 - v_2) + c_2 u_1(v_1 - v_2) \\
&\quad + c_2 v_2(u_1 - u_2) = 0, \quad \text{in } G.
\end{aligned}$$

From these two equations and the periodic property, the following two facts follow:

(1) $\quad b_1(u_1 + u_2) - (a_1 - f) + c_1 v_1 \geqslant \delta_1, \qquad b_2(v_1 + v_2) - (a_2 - g) + c_2 u_1 \geqslant \delta_2,$

by hypothesis (H1) and (2.13);

(2) by the hypothesis (H2) when $i = 1$, $j = 2$ and (2.13), we have

$$c_1 u_2 + c_2 v_2 \leqslant 2 \min\{\delta_1, \delta_2\}.$$

We can readily conclude the following equalities:

$$\begin{aligned}
&\int_G \left[ |\nabla(u_1 - u_2)|^2 + |\nabla(v_1 - v_2)|^2 \right] \mathrm{d}x \, \mathrm{d}t + \int_G \left[ \delta_1(u_1 - u_2)^2 + \delta_2(v_1 - v_2)^2 \right] \mathrm{d}x \, \mathrm{d}t \\
&\qquad - 2 \min\{\delta_1, \delta_2\} \int_G |(u_1 - u_2)| \, |(v_1 - v_2)| \, \mathrm{d}x \, \mathrm{d}t \leqslant 0.
\end{aligned}$$

Hence we finally obtain

$$u_1 = u_2, \quad v_q = v_2, \qquad \text{in } G. \qquad \square$$

Having proved the existence and uniqueness of problem (1.1), we can now prove the existence of an optimal control.

**Theorem 2.4.** *Let $\delta_i$ and $a_i$, $i = 1, 2$, satisfy the hypothesis* (H1). *Then an optimal control does exist for problem* (1.1)–(1.3).

**Proof.** From (2.14), it follows that

$$\sup_{(f,g) \in \boldsymbol{B}_{\delta,T}} J(f, g) < \infty.$$

Let $(f_n, g_n)$ be a maximizing sequence. Then, there exists a subsequence, again denoted as $f_n$ for convenience, so that

$$f_n \to f^*, \quad g_n \to g^*, \quad \text{weakly in } L^2(G) \quad \text{with } (f^*, g^*) \in \boldsymbol{B}_{\delta,T},$$

and

$$u_n(f_n, g_n) \to u^*, \quad v_n(f_n, g_n) \to v^*, \quad \text{strongly in } W_p^{1,0}(G) \text{ and weakly in } W_p^{2,1}(G).$$

Since

$$\begin{cases} u_{nt} - \Delta u_n - u_n[(a_1 - f_n) - b_1 u_n - c_1 v_n] = 0, & \text{in } G, \\ v_{nt} - \Delta v_n - v_n[(a_2 - g_n) - b_2 v_n - c_2 u_n] = 0, & \text{in } G, \\ \dfrac{\partial u_n}{\partial \nu} = \dfrac{\partial v_n}{\partial \nu} = 0, & \text{on } S, \\ u_n(x, 0) = u_n(x, T) \text{ and } v_n(x, 0) = v_n(x, T), & x \in \Omega, \end{cases}$$

we have

$$\int_G \{-u_n \phi_t + \nabla u_n \cdot \nabla \phi - (a_1 - f_n) u_n \phi + b_1 u_n^2 \phi + c_1 u_n v_n \phi\} \, dx \, dt = 0$$

and

$$\int_G \{-v_n \phi_t + \nabla u_n \cdot \nabla \phi - (a_2 - g_n) v_n \phi + b_2 v_n^2 \phi + c_2 u_n v_n \phi\} \, dx \, dt = 0,$$

for any $\phi \in W_p^{1,1}(G) \cap L^\infty(G)$ with $\phi(x, T) = \phi(x, 0)$. Passing to the limits as $n \to \infty$ in the two equalities above, and noting that

$$\int_G f_n u_n \phi \, dx \, dt \to \int_G f^* u^* \phi \, dx \, dt \quad \text{and} \quad \int_G g_n v_n \phi \, dx \, dt \to \int_G g^* v^* \phi \, dx \, dt,$$

for all $\phi \in L^\infty(G)$, we find that $(u^*, v^*)$ is a weak solution of (1.1) with $(f, g)$ replaced by $(f^*, g^*)$. Since $(u^*, v^*) \in W_p^{2,1}(G) \times W_p^{2,1}(G)$, the uniqueness of positive solution of problem (1.1) (Theorem 2.3) implies that

$$u^* = u^*(f^*, g^*) \quad \text{and} \quad v^* = v^*(f^*, g^*).$$

Moreover, we have

$$\begin{aligned} J(f^*, g^*) &= \int_G \{K_1 u^* f^* + K_2 v^* g^* - M_1 f^{*2} - M_2 g^{*2}\} \, dx \, dt \\ &\geqslant \lim_{n \to \infty} \int_G \{K_1 u_n f_n + K_2 v_n g_n\} \, dx \, dt - \liminf_{n \to \infty} \int_G \{M_1 f_n^2 + M_2 g_n^2\} \, dx \, dt \\ &= \limsup_{n \to \infty} J(f^*, g^*) = \sup_{(f,g) \in \boldsymbol{B}_{\delta,T}} J(f, g). \end{aligned}$$

Hence $(f^*, g^*)$ is an optimal control in $\boldsymbol{B}_{\delta,T}$ and this completes the proof. □

## 3. Derivation of the optimal system

In this section we will find some sufficient conditions on $M_i$ and $K_i$ which enable the characterization of an optimal control in terms of a solution of a related parabolic system.

**Lemma 3.1.** *Suppose that $\delta_i$ and $a_i$, $i = 1, 2$, satisfy the conditions* (H1) *and* (H2); *then we have*

$$\frac{u(f + \beta \bar{f}, g + \beta \bar{g}) - u(f, g)}{\beta} \to \xi, \quad \textit{weakly in } W_2^{2,1}(G), \tag{3.1}$$

*and*

$$\frac{v(f+\beta\bar{f},\ g+\beta\bar{g})-v(f,\ g)}{\beta}\to\eta,\quad \textit{weakly in } W_2^{2,1}(G),\tag{3.2}$$

*as* $\beta\to 0$ *for some subsequences, for any given* $(f,\ g)\in \boldsymbol{B}_{\delta,T}$, *and* $(\bar{f},\ \bar{g})\in L^\infty(G)$ *such that* $(f+\beta\bar{f},\ g+\beta\bar{g})\in \boldsymbol{B}_{\delta,T}$. *Furthermore,* $(\xi,\ \eta)$ *is a solution of the problem*

$$\begin{cases}\xi_t-\Delta\xi-[a_1-f-2b_1u(f,\ g)-c_1v(f,\ g)]\xi+c_1\eta u=-\bar{f}u(f,\ g), & \textit{in } G,\\ \dfrac{\partial\xi}{\partial\nu}=0, & \textit{on } S,\\ \xi(x,\ 0)=\xi(x,\ T), & \textit{for all } x\in\Omega,\end{cases}$$
$$\tag{3.3}$$
$$\begin{cases}\eta_t-\Delta\eta-[a_2-g-2b_2v(f,\ g)-c_2u(f,\ g)]\eta+c_2\xi v=-\bar{g}v(f,\ g), & \textit{in } G,\\ \dfrac{\partial\eta}{\partial\nu}=0, & \textit{on } S,\\ \eta(x,\ 0)=\eta(x,\ T), & \textit{for all } x\in\Omega.\end{cases}$$

For the uniqueness of solution to problem (3.3), see Remark 3.2

**Proof.** Let

$$\xi_\beta=\frac{u(f+\beta\bar{f},\ g+\beta\bar{g})-u(f,\ g)}{\beta},\qquad \eta_\beta=\frac{v(f+\beta\bar{f},\ g+\beta\bar{g})-v(f,\ g)}{\beta};$$

then by (1.1) and (1.2), $(\xi_\beta,\ \eta_\beta)$ satisfies

$$\xi_{\beta t}-\Delta\xi_\beta-(a_1-f)\xi_\beta+b_1(\bar{u}+u)\xi_\beta+c_1\bar{v}\xi_\beta+c_1u\eta_\beta=-\bar{f}\bar{u},\quad \text{in } G,\tag{3.4}$$

$$\eta_{\beta t}-\Delta\eta_\beta-(a_2-g)\eta_\beta+b_2(\bar{v}+v)\eta_\beta+c_2\bar{u}\eta_\beta+c_2v\xi_\beta=-\bar{g}\bar{v},\quad \text{in } G,\tag{3.5}$$

$$\frac{\partial\xi_\beta}{\partial\nu}=\frac{\partial\eta_\beta}{\partial\nu}=0,\quad \text{on } S,$$

$$\xi_\beta(x,\ 0)=\xi_\beta(x,\ T)\ \text{ and }\ \eta_\beta(x,\ 0)=\eta_\beta(x,\ T),\quad \text{for all } x\in\Omega.$$

Here, we denote $\bar{u}=u(f+\beta\bar{f},\ g+\beta\bar{g})$, $\bar{v}=v(f+\beta\bar{f},\ g+\beta\bar{g})$. Since $\xi_\beta$ and $\eta_\beta\in W_p^{2,1}(G)$ are periodic with period $T$, we can easily prove by approximation that

$$\int_G\xi_\beta\xi_{\beta t}\,\mathrm{d}x\,\mathrm{d}t=\int_G\eta_\beta\eta_{\beta t}\,\mathrm{d}x\,\mathrm{d}t=\int_G\xi_{\beta t}\Delta\xi_\beta\,\mathrm{d}x\,\mathrm{d}t=\int_G\eta_{\beta t}\Delta\eta_\beta\,\mathrm{d}x\,\mathrm{d}t=0.\tag{3.6}$$

In fact, the equalities are all proved similarly, and we will only prove one of them here. Let $\{r_m\}$ be a sequence of $C^\infty(G)$-periodic functions in $t$ with period $T$, and $\partial r_m/\partial\nu=0$ for all $(x,\ t)\in\partial\Omega\times[0,\ T)$, $m=1,\ 2,\ldots,$ such that

$$r_{m,t}\to\xi_{\beta,t},\quad \Delta r_m\to\Delta\xi_\beta,\qquad \text{in } L^2(G),\quad \text{as } m\to\infty.$$

(Since $\xi_\beta \in W_p^{2,1}(G)$, the existence of such $\{r_m\}$ is given in [4, Chapter 7].) Then we have

$$\int_G \xi_{\beta,t} \Delta \xi_\beta \, \mathrm{d}x \, \mathrm{d}t = \lim_{m\to\infty} \int_G r_{m,t} \Delta r_m \, \mathrm{d}x \, \mathrm{d}t = \lim_{m\to\infty} \int_G -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\nabla r_m|^2 \, \mathrm{d}x \, \mathrm{d}t = 0.$$

Here the second equality is due to the divergence theorem, the last one is due to the periodic property of $r_m$.

Multiplying (3.4), (3.5) by $\xi_\beta$, $\eta_\beta$ respectively and integrating both on $G$, we obtain

$$\int_G |\nabla \xi_\beta|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_G [b_1(\bar{u}+u) + c_1\bar{v} - (a_1 - f)]\xi_\beta^2 \, \mathrm{d}x \, \mathrm{d}t + \int_G c_1 u \xi_\beta \eta_\beta \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\int_G f\bar{u}\xi_\beta \, \mathrm{d}x \, \mathrm{d}t$$

and

$$\int_G |\nabla \eta_\beta|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_G [b_2(\bar{v}+v) + c_2\bar{u} - (a_2 - g)]\eta_\beta b^2 \, \mathrm{d}x \, \mathrm{d}t + \int_G c_2 v \eta_\beta \xi_\beta \, \mathrm{d}x \, \mathrm{d}t$$
$$= -\int_G g\bar{v}\eta_\beta \, \mathrm{d}x \, \mathrm{d}t.$$

From Theorem 2.2 and (H1), we find

$$b_1(\bar{u}+u) - (a_i - f) \geqslant \delta_1, \qquad b_2(\bar{v}+v) - (a_2 - \bar{g}) \geqslant \delta_2. \tag{3.7}$$

Moreover, by (H2) when $i=1$, $j=2$, we have

$$c_1 u + c_2 v \leqslant 2 \min\{\delta_1, \delta_2\}. \tag{3.8}$$

Consequently we obtain the inequality

$$\int_G \left[ |\nabla \xi_\beta|^2 + |\nabla \eta_\beta|^2 + \xi_\beta^2 + \eta_\beta^2 \right] \mathrm{d}x \, \mathrm{d}t \leqslant \text{const.}, \tag{3.9}$$

where the constant is independent of $\beta$.

By moving all the terms of (3.4) except $\xi_{\beta t} - \Delta \xi_\beta$ to the right-hand side and using (3.9), we obtain the following inequality by means of parabolic estimates:

$$\| \xi_\beta \|_{W_2^{2,1}(G)} \leqslant C, \tag{3.10}$$

where the constant $C$ is independent of $\beta$. Similarly, from (3.5) and (3.9), we have

$$\| \eta_\beta \|_{W_2^{2,1}(G)} \leqslant C, \tag{3.11}$$

where $C$ is a constant which is independent of $\beta$.

Consequently there exist subsequences (for convenience denoted again by $\xi_\beta$ and $\eta_\beta$), such that

$$\xi_\beta \to \xi \quad \text{and } \eta_\beta \to \eta,$$

strongly in $W_2^{1,0}(G)$ and weakly in $W_2^{2,1}(G)$.

Moreover, taking limits as $\beta \to \infty$ in (3.4) and (3.5), we conclude that the limit $(\xi, \eta)$ satisfies (3.3). This completes the proof of the lemma. □

**Remark 3.2.** Under the same hypotheses as Lemma 3.1, we can prove as in Theorem 2.3 that problem (3.3) has only one solution, which is in $W_2^{2,1}(G) \times W_2^{2,1}(G)$. Therefore we can actually conclude that (3.1) and (3.2) hold for the full sequence. This uniqueness proof under the hypotheses (H1) and (H2) is nearly the same as in Theorem 2.3, and will thus be omitted here.

**Theorem 3.3.** *Let $p > 1$ be any positive number. Suppose $a_i(x, t)$, $i = 1, 2$, $\boldsymbol{\delta}$ satisfy the hypotheses* (H1) *and* (H2), *and the positive constants $K_i$, $M_i$, $i = 1, 2$, satisfy the hypotheses*

$$\text{(H3)} \qquad M_i \geqslant \frac{K_i \sup_G a_i(x, t)}{2 b_i \delta_i}, \quad \text{for} \quad i = 1, 2.$$

*For any optimal control $(f, g) \in \boldsymbol{B}_{\delta,T}$, let $(u, v)$ be the solution of problem* (1.1) *with*

$$0 < \epsilon_1 \leqslant u \leqslant C_1, \qquad 0 < \epsilon_2 \leqslant v \leqslant C_2,$$

*and suppose $(z, w)$ is a solution of*

$$\begin{cases} z_t + \Delta z + [2ub_1 + c_1 v - a_1 + f] z - c_2 v w = -K_1 f, & \text{in } G, \\ w_t + \Delta w - [2vb_2 + c_2 u - a_2 + g] w - c_1 u z = -K_2 g, & \text{in } G, \\ \dfrac{\partial z}{\partial \nu} = \dfrac{\partial w}{\partial \nu} = 0, & \text{on } S, \\ z(x, 0) = z(x, T) \text{ and } w(x, 0) = w(x, T), & \text{for } x \in \Omega, \end{cases} \tag{3.12}$$

*satisfying*

$$-\frac{c_2 K_2 \hat{a}_2}{\delta_1 b_2 + c_1 \hat{a}_2} \leqslant z \leqslant K_1, \qquad -\frac{c_1 K_1 \hat{a}_1}{\delta_2 b_1 + c_2 \hat{a}_1} \leqslant w \leqslant K_2. \tag{3.13}$$

*Then the optimal control $(f, g)$ satisfies*

$$f = \frac{u(x, t)}{2M_1}(K_1 - z) \quad \text{and} \quad g = \frac{v(x, t)}{2M_2}(K_2 - w), \qquad \text{in } G. \tag{3.14}$$

*Here $u$, $v$, $z$ and $w$ are in $W_p^{2,1}(G)$ and $\epsilon_i$, $C_i$, $i = 1, 2$, are defined by* (2.13).

**Proof.** Theorem 2.4 implies that the conditions of this theorem suffice to insure the existence of an optimal control in $\boldsymbol{B}_{\beta,T}$.

Let $(f, g) \in \boldsymbol{B}_{\delta,T}$ be an optimal control, i.e., there exists a solution $(u, v)$ of the problem (1.1) for $(f, g)$ such that

$$J(f, g) = \sup_{(f', g') \in \boldsymbol{B}_{\delta,T}} J(f', g').$$

For arbitrary $\tilde{f}$, $\tilde{g} \in L^\infty_+(G)$, $\epsilon > 0$, set

$$\bar{f} = \bar{f}_\epsilon = \begin{cases} \tilde{f}, & \text{if } f \leqslant \delta_1 - \epsilon \| \tilde{f} \|_{\infty,G}, \\ 0, & \text{elsewhere;} \end{cases}$$

similarly we define $\bar{g} = \bar{g}_\epsilon$.

For $\beta > 0$ small enough (say $\beta < \epsilon$), such that $(f + \beta\bar{f},\ g + \beta\bar{g}) \in \boldsymbol{B}_{\beta,T}$, the optimality of $(f, g)$ implies that

$$J(f, g) \geqslant J\big(f + \beta\bar{f},\ g + \beta\bar{g}\big), \tag{3.15}$$

that is,

$$\begin{aligned}
&\int_G \big(K_1 uf + K_2 vg - M_1 f^2 - M_2 g^2\big)\, \mathrm{d}x\, \mathrm{d}t \\
&\quad \geqslant \int_G \Big[K_1 u\big(f + \beta\bar{f},\ g + \beta\bar{g}\big)\big(f + \beta\bar{f}\big) + K_2 v\big(f + \beta\bar{f},\ g + \beta\bar{g}\big) - M_1\big(f + \beta\bar{f}\big)^2 \\
&\qquad\qquad - M_2\big(g + \beta\bar{g}\big)^2\Big]\, \mathrm{d}x\, \mathrm{d}t.
\end{aligned}$$

Dividing by $\beta$ and letting $\beta \to 0$, we obtain from Lemma 3.1,

$$\int_G \Big[K_1 \xi f + K_1 u\bar{f} + K_2 \eta g + K_2 v\bar{g} - 2M_1 f\bar{f} - 2M_2 g\bar{g}\Big]\, \mathrm{d}x\, \mathrm{d}t \leqslant 0. \tag{3.16}$$

Since $(z, w)$ is a solution of problem (3.12) satisfying (3.13), we deduce from (3.16), (3.12), (3.3) and integrating by parts that

$$\int_G \big\{\bar{f}_\epsilon[(K_1 - z)u - 2M_1 f] + \bar{g}_\epsilon[(K_2 - w)v - 2M_2 g]\big\}\, \mathrm{d}x\, \mathrm{d}t \leqslant 0.$$

Now, letting $\tilde{g} = 0$, $\epsilon \to 0^+$, and using the same argument as in the proof of [5, Theorem 3.3], we deduce from hypothesis (H3) and the above inequality that

$$f = \frac{K_1 - z}{2M_1} u(x, t), \quad \text{in } G.$$

Similarly, letting $\tilde{f} = 0$, we obtain

$$g = \frac{K_2 - w}{2M_2} v(x, t), \quad \text{in } G.$$

This completes the proof of the theorem. □

**Remark 3.4.** Suppose $(f, g) \in \boldsymbol{B}_{\delta,T}$ is any optimal control, we see from the above theorem that if $(u, v)$ and $(z, w)$ are the unique solutions of problems (1.1) and (3.12), respectively, then

$(u, v, z, w)$ is a solution of the following optimal system:

$$\begin{aligned}
&u_t - \Delta u - a_1 u + \left( b_1 + \frac{K_1 - z}{2M_1} \right) u^2 + c_1 uv = 0, && \text{in } G,\\
&v_t - \Delta v - a_2 v + \left( b_2 + \frac{K_2 - w}{2M_2} \right) v^2 + c_2 uv = 0, && \text{in } G,\\
&z_t + \Delta z - [2b_1 u - a_1 + c_1 v] z + \frac{(K_1 - z)^2}{2M_1} u - c_2 vw = 0, && \text{in } G,\\
&w_t + \Delta w - [2b_2 v - a_2 + c_2 u] w + \frac{(K_2 - w)^2}{2M_2} v - c_1 uz = 0, && \text{in } G,\\
&\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0, && \text{on } S,\\
&u(x, 0) = u(x, T), \quad v(x, 0) = v(x, T), && x \in \Omega,\\
&z(x, 0) = z(x, T), \quad w(x, 0) = w(x, T), && x \in \Omega.
\end{aligned} \tag{3.17}$$

Thus if (3.17) can be solved for $(u, v, w, z)$, then the optimal control $(f, g)$ can be found by using (3.14).

We next prove problem (3.12) indeed has a unique solution satisfying (3.13).

**Theorem 3.5.** *Under the assumptions of Theorem* 3.3, *problem* (3.12) *has a unique solution* $(z, w) \in W_p^{2,1}(G) \times W_p^{2,1}(G)$ *with*

$$-D_1 \equiv -\frac{c_2 K_2 \hat{a}_2}{\delta_1 b_2 + c_1 \hat{a}_2} \leqslant z \leqslant K_1 \quad \textit{and} \quad -D_2 \equiv -\frac{c_1 K_1 \hat{a}_1}{\delta_2 b_1 + c_2 \hat{a}_1} \leqslant w \leqslant K_2.$$

*Here* $(u, v)$ *satisfies* (1.1) *and* (2.13).

**Proof.** We can easily prove that $(-D_1, -D_2)$, $(K_1, K_2)$ are the lower solution and the upper solution of problem (3.12), respectively, in the region $-D_1 \leqslant z \leqslant K_1$, $-D_2 \leqslant w \leqslant K_2$, i.e.,

$$\begin{aligned}
&(-D_1)_t + \Delta(-D_1) + [2b_1 u + c_1 v - a_1 + f] D_1 - c_2 v K_2 \geqslant -K_1 f, && \text{in } G,\\
&(-D_2)_t + \Delta(-D_2) + [2b_2 v + c_2 u - a_2 + g] D_2 - c_1 u K_1 \geqslant -K_2 g, && \text{in } G,\\
&K_{1t} + \Delta K_1 - [2b_1 u + c_1 v - a_1 + f] K_1 - c_2 v(-D_2) \leqslant -K_1 f, && \text{in } G,\\
&K_{2t} + \Delta K_2 - [2b_2 v + c_2 u - a_2 + g] K_2 - c_1 u(-D_1) \leqslant -K_2 g, && \text{in } G.
\end{aligned}$$

To prove the existence of solution for (3.12), we first define

$$(p_0, q_0) = (-D_1, -D_2), \qquad (p_{-1}, q_{-1}) = (K_1, K_2),$$

and $p_i$, $q_i$, $i = 1, 2, 3, \ldots$, to be solutions of

$$\begin{cases} -p_{it} - \delta p_i + [2b_1 u - a_1 + f + c_1 v]\, p_{i-2} = K_1 f - c_2 v q_{i-1}, & \text{in } G, \\ -q_{it} - \Delta q_i + [2b_2 v - a_2 + g + c_2 u]\, q_{i-2} = K_2 g - c_1 u p_{i-1}, & \text{in } G, \\ \dfrac{\partial p_i}{\partial \nu} = \dfrac{\partial q_i}{\partial \nu} = 0, & \text{on } S, \\ p_i(x, 0) = p_i(x, T) \text{ and } q_i(x, 0) = q_i(x, T), & x \in \Omega. \end{cases} \tag{3.18}$$

The existence of solutions for the scalar problems (3.18) are insured by [5, Theorem 2.2]. In fact, if we denote $\tilde{\phi}(x, s) = \phi(x, -s)$ for any function $\phi(x, t)$, then $\tilde{p}_i$ satisfies the parabolic problem

$$\begin{cases} \tilde{p}_{it} - \Delta \tilde{p}_i + \left[2b_1 \tilde{u} - \tilde{a}_1 + \tilde{f} + c_1 \tilde{v}\right] \tilde{p}_{i-1} = K_1 \tilde{f} - c_2 \tilde{v} \tilde{q}_{i-1}, & \text{in } \Omega \times [-T, 0], \\ \dfrac{\partial \tilde{p}_i}{\partial \nu} = 0, & \text{on } \tilde{S} = \partial\Omega \times [-T, 0], \\ \tilde{p}(x, -T) = \tilde{p}(x, 0), & \text{for all } x \in \Omega. \end{cases}$$

[5, Theorem 2.2] implies that the above parabolic problem has a unique solution. Therefore problem (3.18) has a unique solution $p_i(x, s) = \tilde{p}_i(x, -s)$. The same argument applies to $q_i$.

Given a positive number $R$, we define two functions

$$h_1(p, q) = K_1 f - c_2 v q - [2b_1 u - a_1 + f + c_1 v]\, p + Rp, \tag{3.19}$$

$$h_2(p, q) = K_2 g - c_1 u p - [2b_2 v - a_2 + g + c_2 u]\, q + Rq. \tag{3.20}$$

We choose $R$ to be sufficiently large such that $h_1$ and $h_2$ are increasing in $p$ and $q$ respectively in the domain $\epsilon_1 \leqslant u \leqslant C_1$, $\epsilon_2 \leqslant v \leqslant C_2$, $-D_1 \leqslant p \leqslant K_1$ and $-D \leqslant q \leqslant K_2$. Moreover, it is obvious that $h_1$ and $h_2$ are decreasing in $q$ and $p$ respectively in the above domain.

Using $h_1$ and $h_2$, we can rewrite (3.18) as

$$\begin{cases} -p_{it} - \Delta p_i + Rp_i = h_1(p_{i-2}, q_{i-1}), & \text{in } G, \\ -q_{it} - \Delta q_i + Rq_i = h_2(q_{i-2}, p_{i-1},) & \text{in } G, \\ \dfrac{\partial p_i}{\partial \nu} = \dfrac{\partial q_i}{\partial \nu} = 0, & \text{on } S, \\ p_i(x, 0) = p_i(x, T) \text{ and } q_i(x, 0) = q_i(x, T), & x \in \Omega. \end{cases} \tag{3.18$'$}$$

From the monotone properties of $h_1$ and $h_2$, the maximum principle of linear parabolic equations and the fact that $(p_0, q_0)$, $(p_{-1}, q_{-1})$ are lower and upper solutions, we can prove by means of (3.18′) and induction that

$$p_0 \leqslant p_2 \leqslant \cdots \leqslant p_{2i} \leqslant \cdots \leqslant p_{2i+1} \leqslant \cdots \leqslant p_1 \leqslant p_{-1}$$

and

$$q_0 \leqslant q_2 \leqslant \cdots \leqslant q_{2i} \leqslant \cdots \leqslant q_{2i+1} \leqslant \cdots \leqslant q_1 \leqslant q_{-1}.$$

Moreover, we have

$$\| p_i \|_{W_p^{2,1}(G)} \leqslant C \quad \text{and} \quad \| q_i \|_{W_p^{2,1}(G)} \leqslant C,$$

where $C$ is a constant independent of $i$. Consequently, (3.18′) implies that the limits

$$\lim_{r\to\infty} q_{2r}, \qquad \lim_{r\to\infty} q_{2r-1}, \qquad \lim_{r\to\infty} p_{2r}, \qquad \lim_{r\to\infty} p_{2r-1}$$

exist in $W_p^{2,1}(G)$, say $q_*$, $q^*$, $p_*$ and $p^*$, respectively. Moreover, we have $q_* \leqslant q^*$ and $p_* \leqslant p^*$. It remains to prove that $q_* = q^*$ and $p_* = p^*$. Taking the limit as $i \to \infty$ in (3.18′), we find $(q_*, q^*, p_*, p^*)$ satisfies the problem

$$\begin{aligned}
&-p_{*t} - \Delta p_* + [2b_1u - a_1 + f + c_1v]\,p_* + c_2vq^* = K_1 f, && \text{in } G,\\
&-p^*_t - \Delta p^* + [2b_1u - a_1 + f + c_1v]\,p^* + c_2vq_* = K_1 f, && \text{in } G,\\
&-q_{*t} - \Delta q_* + [2b_2v - a_2 + g + c_2u]\,q_* + c_1up^* = K_2 g, && \text{in } G,\\
&-q^*_t - \Delta q^* + [2b_2v - a_2 + g + c_2u]\,q^* + c_1up_* = K_2 g, && \text{in } G,\\
&\frac{\partial p_*}{\partial \nu} = \frac{\partial p^*}{\partial \nu} = \frac{\partial q_*}{\partial \nu} = \frac{\partial q^*}{\partial \nu} = 0, && \text{on } S,\\
&p_*(x, 0) = p_*(x, T), \quad p^*(x, 0) = p^*(x, T), \quad q_*(x, 0) = q_*(x, T),\\
&\quad q^*(x, 0) = q^*(x, T), && x \in \Omega.
\end{aligned} \tag{3.21}$$

Eq. (3.21) consists of actually two separate systems, each with two equations. Moreover, $(p_*, q^*)$ and $(p^*, q_*)$ satisfy the same system of two equations. From hypotheses (H1) and (H2) and the fact

$$2b_1u - a_1 + f + c_1v \geqslant \delta_1 + c_1v \geqslant \delta_1, \qquad 2b_2v - a_2 + g + c_2u \geqslant \delta_2 + c_2u \geqslant \delta_2,$$

we can prove as in Theorem 2.3 that

$$(p^*, q_*) = (p_*, q^*).$$

Hence, we have proved the existence part. The uniqueness of solution in the prescribed range is proved by using the property that $(-D_1, -D_2)$ and $(K_1, K_2)$ are lower and upper solutions of problem (3.18) and by showing

$$p_{2r} \leqslant z \leqslant p_{2r+1} \text{ and } q_{2r} \leqslant w \leqslant q_{2r+1}, \quad \text{for } r = 1, 2, \ldots,$$

with similar arguments. □

## 4. Solution of the optimality system by monotone scheme

In this section we provide an approximation for the solution $(u, v, z, w)$ of problem (3.17). We construct monotone sequences converging from above and below, providing upper and lower estimates for $(u, v, z, w)$. In the case when the limits of upper and lower iterates agree, then the optimal control problem is completely solved. That is, the optimal control is given by

(3.14) in terms of $(u, v, z, w)$, which is calculated iteratively. We will need the following additional conditions:

(H4) $$\frac{\epsilon_i K_i}{2M_i} \leqslant \delta_i \quad \text{for } i, j = 1, 2 \quad \text{and } i \neq j,$$

(H5) $$\frac{c_j \hat{a}_j b_i}{b_j} \leqslant \frac{K_i^2 \delta_i}{K_j M_i}, \quad \text{for } i, j = 1, 2 \quad \text{and } i \neq j,$$

where $\epsilon_1$, $\epsilon_2$ are positive numbers defined by (2.13).

**Remark 4.1.** Under the additional conditions (H4) and (H5), together with (H1)–(H3), we can prove that $(u_0, v_0, p_0, q_0) = (\epsilon_1, \epsilon_2, 0, 0)$ is a lower solution of (3.17). This implies that the proofs of Theorems 3.3 and 3.5 still hold if we replace $(-D_1, -D_2)$ with $(0, 0)$. Then we can use the same arguments to show that the conclusions of Theorems 3.3 and 3.5 are still true. Consequently, there exists one solution $(u, v, z, w)$ of problem (3.14) such that the functions $u$, $v$, $z$ and $w$ are positive.

Assume (H1)–(H5); let

$$(u_0, v_0, p_0, q_0) = (\epsilon_1, \epsilon_2, 0, 0) \quad \text{and } (u_{-1}, v_{-1}, p_{-1}, q_{-1}) = (C_1, C_2, K_1, K_2).$$

(Recall the definition of $C_1$ and $C_2$ in (2.13).) Given a positive number $Q$, we define four functions as follows:

$$h_1(p, u_1, u_2, v) = p[a_1 - 2b_1 u_1 - c_1 v] + \frac{(K_1 - p)^2}{2M_1} u_2 + Qp,$$

$$h_2(q, v_1, v_2, u) = q[a_2 - 2b_2 v_1 - c_2 u] + \frac{(K_2 - q)^2}{2M_2} v_2 + Qq,$$

$$h_3(u, v, p) = u\left[a_1 - \left(b_1 + \frac{K_1}{2M_1}\right)u + \frac{pu}{2M_1} - c_1 v\right] + Qu,$$

$$h_4(v, u, q) = v\left[a_2 - \left(b_2 + \frac{K_2}{2M_2}\right)v + \frac{qv}{2M_2} - c_2 u\right] + Qv.$$

Obviously we can choose $Q$ large enough such that $h_i$, $i = 1, 2, 3, 4$, have the following properties.

(S1) $h_1$ is increasing in $p$ for $p \in [p_0, p_{-1}]$ with fixed $u_1, u_2 \in [u_0, u_{-1}]$ and $v \in [v_0, v_{-1}]$; moreover, $h_1$ is increasing in $u_2$ but decreasing in $u_1$, $v$ with the other variables fixed in the same intervals.

(S2) The properties of $h_2$ in terms of $q$, $v_1$, $v_2$, $u$ are the same as $h_1$ in terms of $p$, $u_1$, $u_2$, $v$, respectively.

(S3) $h_3$ is increasing in $u$ for $u \in [u_0, u_{-1}]$ with fixed $p \in [p_0, p_{-1}]$ and $v \in [v_0, v_{-1}]$; moreover, it is increasing in $p$ and decreasing in $v$ with the other variables fixed in the same intervals.

(S4) The properties of $h_4$ in terms of $v$, $u$, $q$ are the same as $h_3$ in terms of $u$, $v$, $p$, respectively.

We can readily verify that $(u_0, v_0, p_0, q_0)$ and $(u_{-1}, v_{-1}, p_{-1}, q_{-1})$ satisfy

$$u_{-1t} - \Delta u_{-1} + Qu_{-1} \geqslant h_3(u_{-1}, v_0, p_{-1}), \quad \text{in } G, \tag{4.1}$$

$$v_{-1t} - \Delta v_{-1} + Qv_{-1} \geqslant h_4(v_{-1}, u_0, q_{-1}), \quad \text{in } G, \tag{4.2}$$

$$p_{-1t} + \Delta p_{-1} - Qp_{-1} \leqslant -h_1(p_{-1}, u_0, u_{-1}, v_0) + c_2 v_0 q_0, \quad \text{in } G, \tag{4.3}$$

$$q_{-1t} + \Delta q_{-1} - Qq_{-1} \leqslant -h_2(q_{-1}, v_0, v_{-1}, u_0) + c_1 u_0 p_0, \quad \text{in } G, \tag{4.4}$$

$$p_{0t} + \Delta p_0 - Qp_0 \geqslant -h_1(p_0, u_{-1}, u_0, v_{-1}) + c_2 v_{-1} q_{-1}, \quad \text{in } G, \tag{4.5}$$

$$q_{0t} + \Delta q_0 - Qq_0 \geqslant -h_2(q_0, v_{-1}, v_0, u_{-1}) + c_1 u_{-1} p_{-1}, \quad \text{in } G, \tag{4.6}$$

$$u_{0t} - \Delta u_0 + Qu_0 \leqslant h_3(u_0, v_{-1}, p_0), \quad \text{in } G, \tag{4.7}$$

$$v_{0t} - \Delta v_0 + Qv_0 \leqslant h_4(v_0, u_{-1}, q_0), \quad \text{in } G. \tag{4.8}$$

Inequalities (4.1)–(4.4) can be readily verified using (H1). We next show that (4.5) holds. Since $p_0 = 0$, proving (4.5) is equivalent to proving the inequality

$$0 \geqslant -\left(\frac{K_1^2}{2M_1}\right)\epsilon_1 + c_2\left(\frac{\hat{a}_2}{b_2}\right)K_2.$$

From (H1), we have

$$\epsilon_1 = \frac{\tilde{a}_1 - c_1(\hat{a}_2/b_2) - \delta_1}{b_1} \geqslant \frac{2\delta_1}{b_1}.$$

Thus, in order to prove (4.5), we only need to show

$$\frac{K_1^2\delta_1}{M_1K_2} \geqslant \frac{c_2 a_2 b_1}{b_2},$$

which is our hypothesis (H5). Inequality (4.6) is completely analogous to (4.5). Similarly, using (H1) and (H4), we can prove (4.7) and (4.8).

Now, we inductively define sequences of the functions $u_i$, $v_i$, $p_i$ and $q_i$ for $i = 1, 2, \ldots,$ as solutions of the following scalar problems:

$$\begin{cases} u_{it} - \Delta u_i + Qu_i = h_3(u_{i-2}, v_{i-1}, p_{i-2}), & \text{in } G, \\ \dfrac{\partial u_i}{\partial \nu} = 0, & \text{on } S, \\ u_i(x, 0) = u_i(x, T), & \text{for } x \in \Omega, \end{cases} \tag{4.9}$$

$$\begin{cases} v_{it} - \Delta v_i + Qv_i = h_4(v_{i-2}, u_{i-1}, q_{i-2}), & \text{in } G, \\ \dfrac{\partial v_i}{\partial \nu} = 0, & \text{on } S, \\ v_i(x, 0) = v_i(x, T), & \text{for } x \in \Omega, \end{cases} \tag{4.10}$$

$$\begin{cases} p_{it} + \Delta p_i - Qp_i = -h_1(p_{i-2}, u_{i-1}, u_{i-2}, v_{i-1}) + c_2 v_{i-1} q_{i-1}, & \text{in } G, \\ \dfrac{\partial p_i}{\partial \nu} = 0, & \text{on } S, \\ p_i(x, 0) = p_i(x, T), & x \in \Omega, \end{cases} \tag{4.11}$$

$$\begin{cases} q_{it} + \Delta q_i - Qq_i = -h_2(q_{i-2}, v_{i-1}, v_{i-2}, u_{i-1}) + c_1 u_{i-1} p_{i-1}, & \text{in } G, \\ \dfrac{\partial q_i}{\partial \nu} = 0, & \text{on } S, \\ q_i(x, 0) = q_i(x, T), & \text{for } x \in \Omega. \end{cases} \tag{4.12}$$

The existence of solutions follows from [5, Theorem 2.2]. By using the induction argument, the monotone properties of $h_f$, $i = 1, 2, 3, 4$, and the maximum principle, we can show that

$$\begin{aligned} &u_0 \leqslant u_2 \leqslant \cdots \leqslant u_{2i} \leqslant \cdots \leqslant u_{2i-1} \leqslant \cdots \leqslant u_1 \leqslant u_{-1}, \\ &v_0 \leqslant v_2 \leqslant \cdots \leqslant v_{2i} \leqslant \cdots \leqslant v_{2i-1} \leqslant \cdots \leqslant v_1 \leqslant v_{-1}, \\ &p_0 \leqslant p_2 \leqslant \cdots \leqslant p_{2i} \leqslant \cdots \leqslant p_{2i-1} \leqslant \cdots \leqslant p_1 \leqslant p_{-1}, \\ &q_0 \leqslant q_2 \leqslant \cdots \leqslant q_{2i} \leqslant \cdots \leqslant q_{2i-1} \leqslant \cdots \leqslant q_1 \leqslant q_{-1}. \end{aligned} \tag{4.13}$$

In fact, we first observe that $u_0 < u_{-1}$ in $G$. From (4.1) and (4.9), we verify that

$$\begin{aligned} &(u_{-1} - u_1)_t - \Delta(u_{-1} - u_1) + Q(u_{-1} - u_1) \\ &\quad \geqslant h_3(u_{-1}, v_0, p_{-1}) - h_3(u_{-1}, v_0, p_{-1}) = 0, \quad \text{in } G. \end{aligned} \tag{4.14}$$

Thus by [5, Lemma 2.3], we have

$$u_{-1} \geqslant u_1, \quad \text{in } G. \tag{4.15}$$

By the same reason we can obtain

$$u_0 \leqslant u_2 \leqslant u_1 \leqslant u_{-1}, \qquad v_0 \leqslant v_2 \leqslant v_1 \leqslant v_{-1}, \qquad p_0 \leqslant p_2 \leqslant p_1 \leqslant p_{-1}, \qquad q_0 \leqslant q_2 \leqslant q_1 \leqslant q_{-1}. \tag{4.16}$$

Suppose we have proved

$$\begin{aligned} &u_0 \leqslant u_2 \leqslant \cdots \leqslant u_{2r} \leqslant u_{2r-1} \leqslant \cdots \leqslant u_1 \leqslant u_{-1}, \\ &v_0 \leqslant v_2 \leqslant \cdots \leqslant v_{2r} \leqslant v_{2r-1} \leqslant \cdots \leqslant v_1 \leqslant v_{-1}, \\ &p_0 \leqslant p_2 \leqslant \cdots \leqslant p_{2r} \leqslant p_{2r-1} \leqslant \cdots \leqslant p_1 \leqslant p_{-1}, \\ &q_0 \leqslant q_2 \leqslant \cdots \leqslant q_{2r} \leqslant q_{2r-1} \leqslant \cdots \leqslant q_1 \leqslant q_{-1}. \end{aligned} \tag{4.17}$$

From (4.9), we obtain

$$\begin{aligned} &(u_{2r+1} - u_{2r})_t - \Delta(u_{2r+1} - u_{2r}) + Q(u_{2r+1} - u_{2r}) \\ &\quad = h_3(u_{2r-1}, v_{2r}, p_{2r-1}) - h_3(u_{2r-2}, v_{2r-1}, p_{2r-2}) \geqslant 0, \end{aligned}$$

where the last inequality is a consequence of (S3) and (4.17).

Hence, [5, Lemma 2.3] implies that

$$u_{2r+1} \geqslant u_{2r}, \quad \text{in } G. \tag{4.18}$$

Similarly, we have

$$v_{2r+1} \geqslant v_{2r}, \quad \text{in } G. \tag{4.19}$$

Moreover, from (4.9), (4.17), (4.19) and (S3), we easily deduce that

$$\begin{aligned}
&(u_{2r}-u_{2r+2})_t - \Delta(u_{2r}-u_{2r+2}) + Q(u_{2r}-u_{2r+2}) \leqslant 0, && \text{in } G,\\
&(u_{2r+1}-u_{2r-1})_t - \Delta(u_{2r+1}-u_{2r-1}) + Q(u_{2r+1}-u_{2r-1}) \leqslant 0, && \text{in } G,\\
&(u_{2r+2}-u_{2r+1})_t - \Delta(u_{2r+2}-u_{2r+1}) + Q(u_{2r+2}-u_{2r+1}) \leqslant 0, && \text{in } G.
\end{aligned}$$

Hence, [5, Lemma 2.3] again implies that

$$u_{2r} \leqslant u_{2r+2} \leqslant u_{2r+1} \leqslant u_{2r-1}, \quad \text{in } G. \tag{4.20}$$

Moreover, we can deduce the same inequalities as in (4.20) for $v$, $p$, $q$. (For more details on similar procedures, see [10, Chapter 5].) Hence we have the following theorem.

**Theorem 4.2.** *Assume hypotheses* (H1)–(H5). *The sequences of functions $u_i$, $v_i$, $p_i$ and $q_i$ defined above satisfy the order relation* (4.17) *for all positive integer $r$ and $(x, t) \in G$. Moreover, any solution $(u, v, z, w)$ of problem* (3.17) *with the properties*

$$u_0 \leqslant u \leqslant u_{-1}, \quad v_0 \leqslant v \leqslant v_{-1}, \quad p_0 \leqslant z \leqslant p_{-1}, \quad q_0 \leqslant w \leqslant q_{-1}, \quad \text{in } G, \tag{4.21}$$

*must satisfy the inequalities*

$$u_{2i} \leqslant u \leqslant u_{2i-1}, \quad v_{2i} \leqslant v \leqslant v_{2i-1}, \quad p_{2i} \leqslant z \leqslant p_{2i-1}, \quad q_{2i} \leqslant w \leqslant q_{2i-1}, \quad \text{in } G. \tag{4.22}$$

*for any positive integer $i$.*

**Proof.** It remains only to prove the second part of this theorem. From (4.9), (4.21) and the monotone property (S3), we have

$$(u-u_1)_t - \Delta(u-u_1) + Q(u-u_1) = h_3(u, v, p) - h_3(u_{-1}, v_0, p_{-1}) \leqslant 0, \quad \text{in } G.$$

Thus [5, Lemma 2.3] implies that $u \leqslant u_1$ in $G$. As above, we can use induction and the monotone properties of $h_i$, $i = 1, 2, 3, 4$, to prove the other inequalities of (4.22). □

**Remark 4.3.** From Theorems 3.5 and 4.2, we find that if

$$\lim_{r\to\infty} u_{2i} = \lim_{r\to\infty} u_{2i-1}, \qquad \lim_{r\to\infty} v_{2i} = \lim_{r\to\infty} v_{2i-1}, \qquad \lim_{r\to\infty} p_{2i} = \lim_{r\to\infty} p_{2i-1},$$
$$\lim_{r\to\infty} q_{2i} = \lim_{r\to\infty} q_{2i-1},$$

then the optimal control problem described in Section 1 is completely solved. This had been explained in the beginning of this section (cf. also [10, Chapter 5]).

## 5. Example

In problems (1.1) and (1.2), let $\Omega = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$ and $G = \Omega \times [0, 2\pi]$. Define $a_1 = [\frac{1}{4}(x^2+y^2)]\cos t + 16$, $b_1 = 4$, $c_1 = 0.4$, $a_2 = \sin x\pi \sin y\pi \sin t + 25$, $b_2 = 6$, $c_2 = 0.5$, $K_1 = 8$, $K_2 = 7$, $M_1 = 4$, $M_2 = 5$. We thus have $\tilde{a}_1 = 16 - \frac{1}{4}$, $\hat{a}_1 = 16 + \frac{1}{4}$, $\tilde{a}_2 = 24$ and $\hat{a}_2 = 26$.

Choosing $\boldsymbol{\delta} = (\delta_1, \delta_2) = (11, 17)$, we can easily verify that hypotheses (H1)–(H5) are satisfied. For example, considering hypothesis (H4), with $i = 1$, $j = 2$, we have

$$\epsilon_1 = \frac{\tilde{a}_1 - c_1(\hat{a}_2/b_2) - \delta_1}{b_1} = \tfrac{19}{16} - 0.4 < 2;$$

thus,

$$\frac{\epsilon_i K_i}{2M_1} < 2 < \delta_1 = 11,$$

i.e., (H3) holds for $i = 1$ and $j = 2$. Similarly, (H4) holds for $i = 2$ and $j = 1$.

**Remark 5.1.** Let $A_1(x, t)$ and $A_2(x, t)$ be given continuous $t$-periodic functions in $G = \Omega \times (-\infty, \infty)$, where $\Omega$ is any bounded domain with $C^2$ boundary. Consider problem (1.1), (1.2) with fixed $c_i$, $b_i$, $M_i$ and $K_i$ for $i = 1, 2$. From the previous example, we see that we can always find a large enough constant $B$ and $\boldsymbol{\delta}$ such that if we define $a_1 = A_i + B$, $i = 1, 2$, then the hypotheses (H1)–(H4) are readily satisfied. Consequently, our results are applicable to a large family of problems.

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